Tensor

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The term **tensor** has slightly different meanings in mathematics and physics. In the mathematical fields of multilinear algebra and differential geometry, a tensor is a multilinear function. In physics and engineering, the same term usually means what a mathematician would call a tensor field: an association of a different (mathematical) tensor with each point of a geometric space, varying continuously with position.

Contents

- 1 History
- 2 Two usages of 'tensor'
  - 2.1 Mathematical
  - 2.2 Physical - tensor fields
- 3 Tensor rank
- 4 Tensor valence
- 5 Importance and applications
- 6 The choice of approach
- 7 Examples
  - 7.1 Physical examples
  - 7.2 Mathematical examples
- 8 Approaches, in detail
- 9 Tensor densities
- 10 See also
  - 10.1 Notation
  - 10.2 Foundational
  - 10.3 Applications
- 11 External links
- 12 Historic references
- 13 References
- 14 Tensor software

History

The word *tensor* was introduced in 1846 by William Rowan Hamilton[1] to describe the norm operation in a certain type of algebraic system (eventually known as a Clifford algebra). The word was used in its current meaning by Woldemar Voigt in 1899.

Tensor calculus was developed around 1890 by Gregorio Ricci-Curbastro under the title *absolute differential calculus*, and was made accessible to many mathematicians by the publication of Tullio Levi-Civita's 1900 classic text of the same name (in Italian; translations followed). In the 20th century, the subject came to be known as *tensor analysis*, and achieved broader acceptance with the introduction of Einstein's theory of general relativity, around 1915.

General relativity is formulated completely in the language of tensors. Einstein had learned about them, with great difficulty, from the geometer Marcel Grossmann,[2] or perhaps from Levi-Civita himself. Tensors are used also in other fields such as continuum mechanics.

Two usages of 'tensor'

Mathematical

In mathematics, a **tensor** is (in an informal sense) a generalized linear 'quantity' or 'geometrical entity' that can be expressed as a multi-dimensional array relative to a choice of basis of the particular space on which it is defined. The intuition underlying the tensor concept is inherently geometrical: as an object in and of itself, a tensor is *independent of any chosen frame of reference*. However, in the modern treatment, tensor theory is best regarded as a topic in multilinear algebra. Engineering applications do not usually require the full, general theory, but theoretical physics now does.

For example, the Euclidean inner product (dot product) — a real-valued function of two vectors that is linear in each — is a mathematical tensor. Similarly, on a smooth curved surface such as a torus, the metric tensor (field) essentially defines a different inner product of tangent vectors at each point of the surface. Just as a linear transformation can be represented as a matrix of numbers with respect to given vector bases, so a tensor can be written as an organized collection of numbers. In physics, the numbers may be obtained as physical quantities that depend on a basis, and the collection is determined to be a tensor if the quantities transform appropriately under change of basis.

Physical - tensor fields

Many mathematical structures informally called 'tensors' are actually 'tensor fields' — an abstraction of tensors to field, wherein tensorial quantities vary from point to point. Differential equations posed in terms of tensor quantities are basic to modern mathematical physics, so that methods of differential calculus are also applied to tensors.

Tensor rank

The **rank** of a particular tensor is the number of array indices required to describe such a quantity. For example in classical mechanics, mass, temperature, and other scalar quantities are tensors of rank 0; but force, momentum and other vector-like quantities are tensors of rank 1. The novel aspects of tensor theory are seen from rank 2 onwards. A linear transformation such as an anisotropic relationship (relativistic mass) between force and acceleration vectors is a tensor of rank 2. There exists a similar relationship between temperature in different reference frames in relativistic thermodynamics.[3]

Tensor valence

In physical applications, array indices are distinguished by being contravariant (superscripts) or covariant (subscripts), depending upon the type of transformation properties. The **valence** of a particular tensor is the number and type of array indices; tensors with the same rank but different valence are not, in general, identical. However, any given covariant index can be transformed into a contravariant one, and vice versa, by applying the metric tensor. This operation is generally known as **raising** or **lowering** indices.

Importance and applications

Tensors are important in physics and engineering. In the field of diffusion tensor imaging, for instance, a tensor quantity that expresses the differential permeability of organs to water in varying directions is used to produce scans of the brain; in this technique tensors are in effect made visible. Perhaps the most important engineering examples are the stress tensor and strain tensor, which are both 2nd rank tensors, and are related in a general linear material by a fourth rank elasticity tensor.

Specifically, a 2nd rank tensor quantifying stress in a 3-dimensional/solid object has components which can be conveniently represented as a 3x3 array. The three Cartesian faces of a cube-shaped infinitesimal volume segment of the solid are each subject to some given force. The force's vector components are also three in number (being in three-space). Thus, 3x3, or 9 components are required to describe the stress at this cube-shaped infinitesimal segment (which may now be treated as a point). Within the bounds of this solid is a whole mass of varying stress quantities, each requiring 9 quantities to describe. Thus, the need for a 2nd order tensor is produced.

While tensors can be represented by multi-dimensional arrays of components, the point of having a tensor *theory* is to explain further implications of saying that a quantity is a *tensor*, beyond specifying that it requires a number of indexed components. In particular, tensors behave in specific ways under coordinate transformations. The abstract theory of tensors is a branch of linear algebra, now called multilinear algebra.

The choice of approach

There are two ways of approaching the definition of tensors:

- The usual physics way of defining tensors, in terms of objects whose components transform according to certain rules, introducing the ideas of covariant or contravariant transformations.

- The usual mathematics way, which involves defining certain vector spaces and not fixing any coordinate systems until bases are introduced when needed. Covariant vectors, for instance, can also be described as one-forms, or as the elements of the dual space to the contravariant vectors.

Physicists and engineers are among the first to recognise that vectors and tensors have a physical significance as entities, which goes beyond the (often arbitrary) co-ordinate system in which their components are enumerated. Similarly, mathematicians find there are some tensor relations which are more conveniently derived in a co-ordinate notation.

Examples

Physical examples

As a simple example, consider a ship in the water. We want to describe its response to an applied force. Force is a vector, and the ship will respond with an acceleration, which is also a vector. The relationship between force and acceleration is linear in classical mechanics. Such a relationship is described by a rank two tensor of type (1,1) (that is to say, here it transforms a plane vector into another such vector). The tensor can be represented as a matrix which when multiplied by a vector results in another vector. Just as the numbers which represent a vector will change if one changes the coordinate system, the numbers in the matrix that represents the tensor will also change when the coordinate system is changed.

In engineering, the stresses inside a solid body or fluid are also described by a tensor; the word "tensor" is Latin for something that stretches, i.e., causes tension. If a particular surface element inside the material is singled out, the material on one side of the surface will apply a force on the other side. In general, this force will not be orthogonal to the surface, but it will depend on the orientation of the surface in a linear manner. This is described by a tensor of type (2,0), in linear elasticity, or more precisely by a tensor *field* of type (2,0) since the stresses may change from point to point.

Mathematical examples

Some well-known examples of tensors in differential geometry are quadratic forms, such as metric tensors, and the curvature tensor.

Formally speaking, a tensor has a particular type according to the construction with tensor products that give rise to it. For computational purposes, it may be expressed as the sequence of values represented by a function with a tuple-valued domain and a scalar valued range. Domain values are tuples of counting numbers, and these numbers are called indices. For example, a rank 3 tensor might have dimensions 2, 5, and 7. Here, the indices range from «1, 1, 1» through «2, 5, 7»; thus the tensor would have one value at «1, 1, 1», another at «1, 1, 2», and so on for a total of 70 values. As a special case, (finite-dimensional) vectors may be expressed as a sequence of values represented by a function with a scalar valued domain and a scalar valued range; the number of distinct indices is the dimension of the vector. Using this approach, the rank 3 tensor of dimension (2,5,7) can be represented as a 3-dimensional array of size $2 \times 5 \times 7$. In this usage, the number of "dimensions" comprising the array is equivalent to the "rank" of the tensor, and the dimensions of the tensor are equivalent to the "size" of each array dimension.

A tensor field associates a tensor value with every point on a manifold. Thus, instead of simply having 70 values as indicated in the above example, for a rank 3 tensor field with dimensions «2, 5, 7»; every point in the space would have 70 values associated with it. In other words, a tensor field means there's some tensor-valued function which has, for example, Euclidean space as its domain.

Approaches, in detail

There are ***equivalent*** approaches to visualizing and working with tensors; that the content is actually the same may only become apparent with some familiarity with the material.

- The **classical approach**

  The classical approach views tensors as multidimensional arrays that are *n*-dimensional generalizations of scalars, 1-dimensional vectors and 2-dimensional matrices. The "components" of the tensor are the values in the array. This idea can then be further generalized to tensor fields, where the elements of the tensor are functions, or even differentials.

  However, to count as a tensor, the arrays need to transform correctly when the reference co-ordinate system is changed. This transformation is a generalisation of the relationship which holds for vector components, and is similarly an expression of the independence of the underlying entity from the reference frame in which it is expressed.

- The **modern approach**

The modern (component-free) approach views tensors initially as abstract objects, expressing some definite type of multi-linear concept. Their well-known properties can be derived from their definitions, as linear maps or more generally; and the rules for manipulations of tensors arise as an extension of linear algebra to multilinear algebra. This treatment has attempted to replace the component-based treatment for advanced study, in the way that the more modern component-free treatment of vectors replaces the traditional component-based treatment after the component-based treatment has been used to provide an elementary motivation for the concept of a vector. You could say that the slogan is 'tensors are elements of some tensor space'. Nevertheless, a component-free approach has not become fully popular, owing to the difficulties involved with giving a geometrical interpretation to higher-rank tensors.

- The **intermediate treatment of tensors** article attempts to bridge the two extremes, and to show their relationships.

In the end the same computational content is expressed, both ways. See glossary of tensor theory for a listing of technical terms.

Tensor densities

It is also possible for a tensor field to have a "density". A tensor with density *r* transforms as an ordinary tensor under coordinate transformations, except that it is also multiplied by the determinant of the Jacobian to the r^{th} power. Invariantly, in the language of multilinear algebra, one can think of tensor densities as multilinear maps taking their values in the (1-dimensional) space of *n*-forms (where *n* is the dimension of the space), as opposed to taking their values in just **R**. Higher "weights" then just correspond to taking additional tensor products with this space in the range. In the language of vector bundles, the determinant bundle of the tangent bundle is a line bundle that can be used to 'twist' other bundles *r* times.

See also

- Glossary of tensor theory

Notation

- Abstract index notation
- Einstein notation
- Voigt notation
- Mandel notation
- Diagrammatic notation
- Raising and lowering indices

Foundational

- Contravariant
- Covariant
- Fibre bundle
- One-form
- Tensor field
- Tensor product

Applications

- Absolute differentiation
- Application of tensor theory in engineering
- Application of tensor theory in physics
- Curvature
- Einstein field equations
- Fluid mechanics
- Riemannian geometry
- Tensor derivative
- Structure Tensor

External links

- An Introduction to Tensors for Students of Physics and Engineering, released by NASA
- A discussion of the various approaches to teaching tensors, and recommendations of textbooks

- A thread discussing basic and in depth definitions as well as various examples
- A Quick Introduction to Tensor Analysis by R. A. Sharipov.

Historic references

1. ^ William Rowan Hamilton, *On some Extensions of Quaternions* [1]
2. ^ Abraham Pais *Subtle is the Lord*
3. ^ Tolman, Richard (1934). *Thermodynamics and Cosmology*. Oxford: Oxford University Press, Oxford.

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Tensor software

- FTensor is a high performance tensor library written in C++.
- GRTensorII is a computer algebra package for performing calculations in the general area of differential geometry. GRTensor II is not a stand alone package, the program runs with all versions of Maple V Release 3 through Maple 9.5. A limited version (GRTensorM) has been ported to Mathematica.
- MathTensor is a tensor analysis system written for the Mathematica system. It provides more than 250 functions and objects for elementary and advanced users.
- Tensors in Physics is a tensor package written for the Mathematica system. It provides many functions relevant for General Relativity calculations in general Riemann-Cartan geometries.
- maxima is a free computer algebra system which should be usable for making tensor algebra calculations
  - tensors in maxima
- Ricci is a system for Mathematica 2.x and later for doing basic tensor analysis, available for free.
- Tela is a software package similar to Matlab and Octave, but designed specifically for tensors.
- Tensor Toolbox Multilinear algebra MATLAB software.
- TTC Tools of Tensor Calculus is a Mathematica package for doing tensor and exterior calculus on differentiable manifolds.
- EDC and RGTC "Exterior Differential Calculus" and "Riemannian Geometry & Tensor Calculus" are Mathematica packages for tensor calculus especially designed but not only for general relativity.