Jacobian

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In vector calculus, the **Jacobian** is shorthand for either the **Jacobian matrix** or its determinant, the **Jacobian determinant**.

In algebraic geometry the **Jacobian** of a curve means the Jacobian variety: a group variety associated to the curve, in which the curve can be embedded.

These concepts are all named after the mathematician Carl Gustav Jacobi. The term "Jacobian" is normally pronounced [jaˈkobiən], but can also be pronounced [ʤəˈkobiən].

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Jacobian matrix

The **Jacobian matrix** is the matrix of all first-order partial derivatives of a vector-valued function. Its importance lies in the fact that it represents the best linear approximation to a differentiable function near a given point. In this sense, the Jacobian is akin to a derivative of a multivariate function. For $n > 1$, the derivative of a numerical function must be matrix-valued, or a partial derivative.

Suppose $F : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is a function from Euclidean n-space to Euclidean m-space. Such a function is given by m real-valued component functions, $y_1(x_1,...,x_n)$, ..., $y_m(x_1,...,x_n)$. The partial derivatives of all these functions (if they exist) can be organized in an m-by-n matrix, the Jacobian matrix of F, as follows:

$$\begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}.$$

This matrix is denoted by

$$J_F(x_1,\ldots,x_n) \text{ or by } \frac{\partial(y_1,\ldots,y_m)}{\partial(x_1,\ldots,x_n)}.$$

The ith row of this matrix is given by the transpose of the gradient of the function y_i for $i = 1,...,m$.

If $\mathbf{p}$ is a point in $\mathbf{R}^n$ and F is differentiable at $\mathbf{p}$, then its derivative is given by $J_F(\mathbf{p})$ (and this is the easiest way to compute the derivative). In this case, the linear map described by $J_F(\mathbf{p})$ is the best linear approximation of F near the point $\mathbf{p}$, in the sense that

$$F(\mathbf{x}) \approx F(\mathbf{p}) + J_F(\mathbf{p})\cdot(\mathbf{x}-\mathbf{p})$$

for $\mathbf{x}$ close to $\mathbf{p}$.

Note that the Jacobian of the gradient is the Hessian matrix.

Examples

The transformation from spherical coordinates to Cartesian coordinates is given by the function $F : \mathbf{R} \times [0,\pi] \times [0,2\pi] \to \mathbf{R}^3$ with components:

$$\begin{aligned} x_1 &= r\sin\phi\cos\theta \\ x_2 &= r\sin\phi\sin\theta \\ x_3 &= r\cos\phi \end{aligned}$$

The Jacobian matrix for this coordinate change is

$$J_F(r,\phi,\theta) = \begin{bmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \phi} & \frac{\partial x_1}{\partial \theta} \\ \frac{\partial x_2}{\partial r} & \frac{\partial x_2}{\partial \phi} & \frac{\partial x_2}{\partial \theta} \\ \frac{\partial x_3}{\partial r} & \frac{\partial x_3}{\partial \phi} & \frac{\partial x_3}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\phi & -r\sin\phi & 0 \end{bmatrix}.$$

The Jacobian matrix of the function $F : \mathbf{R}^3 \to \mathbf{R}^4$ with components

$$\begin{aligned} y_1 &= x_1 \\ y_2 &= 5x_3 \\ y_3 &= 4x_2^2 - 2x_3 \\ y_4 &= x_3\sin(x_1) \end{aligned}$$

is

$$J_F(x_1,x_2,x_3) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \\ \frac{\partial y_4}{\partial x_1} & \frac{\partial y_4}{\partial x_2} & \frac{\partial y_4}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 8x_2 & -2 \\ x_3\cos(x_1) & 0 & \sin(x_1) \end{bmatrix}.$$

This example shows that the Jacobian need not be a square matrix.

In dynamical systems

Consider a dynamical system of the form $x' = F(x)$, with $F : \mathbf{R}^n \to \mathbf{R}^n$. If $F(x_0) = 0$, then x_0 is a stationary point. The behavior of the system near a stationary point can often be determined by the eigenvalues of $J_F(x_0)$, the Jacobian of F at the stationary point.[1]

Jacobian determinant

If $m = n$, then F is a function from n-space to n-space and the Jacobian matrix is a square matrix. We can then form its determinant, known as the **Jacobian determinant**. The Jacobian determinant is also called the "Jacobian" in some sources.

The Jacobian determinant at a given point gives important information about the behavior of F near that point. For instance, the continuously differentiable function F is invertible near **p** if the Jacobian determinant at **p** is non-zero. This is the inverse function theorem. Furthermore, if the Jacobian determinant at **p** is positive, then F preserves orientation near **p**; if it is negative, F reverses orientation. The absolute value of the Jacobian determinant at **p** gives us the factor by which the function F expands or shrinks volumes near **p**; this is why it occurs in the general substitution rule.

Example

The Jacobian determinant of the function $F : \mathbf{R}^3 \to \mathbf{R}^3$ with components

$$\begin{aligned} y_1 &= 5x_2 \\ y_2 &= 4x_1^2 - 2\sin(x_2x_3) \\ y_3 &= x_2x_3 \end{aligned}$$

is

$$\begin{vmatrix} 0 & 5 & 0 \\ 8x_1 & -2x_3\cos(x_2x_3) & -2x_2\cos(x_2x_3) \\ 0 & x_3 & x_2 \end{vmatrix} = -8x_1 \cdot \begin{vmatrix} 5 & 0 \\ x_3 & x_2 \end{vmatrix} = -40x_1x_2.$$

From this we see that *F* reverses orientation near those points where x_1 and x_2 have the same sign; the function is locally invertible everywhere except near points where $x_1 = 0$ or $x_2 = 0$. If you start with a tiny object around the point (1,1,1) and apply *F* to that object, you will get an object set with about 40 times the volume of the original one.

Uses

The Jacobian determinant is used when making a change of variables when integrating a function over its domain. To accommodate for the change of coordinates the Jacobian determinant arises as a multiplicative factor within the integral. Normally it is required that the change of coordinates is done in a manner which maintains an injectivity between the coordinates that determine the domain. The Jacobian determinant, as a result, is usually well defined.

See also

- Pushforward (differential)
- Hessian matrix

References

1. ^ D.K. Arrowsmith and C.M. Place, *Dynamical Systems*, Section 3.3, Chapman & Hall, London, 1992. ISBN 0-412-39080-9.

External links

- Ian Craw's Undergraduate Teaching Page An easy to understand explanation of Jacobians
- Mathworld A more technical explanation of Jacobians

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