

Dirac equation

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In physics, the **Dirac equation** is a relativistic quantum mechanical wave equation formulated by British physicist Paul Dirac in 1928 and provides a description of elementary spin-½ particles, such as electrons, consistent with both the principles of quantum mechanics and the theory of special relativity. The equation demands the existence of antiparticles and actually predated their experimental discovery, making the discovery of the positron, the antiparticle of the electron, one of the greatest triumphs of modern theoretical physics.

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Details

The Dirac Equation in the form originally proposed by him is:

$$\left(\beta mc^2 + \sum_{k=1}^3 \alpha_k p_k c \right) \psi(\mathbf{x}, t) = i\hbar \frac{\partial \psi}{\partial t}(\mathbf{x}, t)$$

where

m is the rest mass of the electron,
 c is the speed of light,
 p is the momentum operator,
 \hbar is the reduced Planck's constant,
 \mathbf{x} and t are the space and time coordinates.

The new elements in this equation are the 4x4 matrices α_k and β , and the four-component wavefunction ψ . The matrices are all Hermitian and have squares equal to the identity matrix, and they all mutually anticommute:

$$\begin{aligned} \alpha_i \alpha_j &= -\alpha_j \alpha_i, \\ \alpha_i \beta &= -\beta \alpha_i, \end{aligned}$$

where i and j are distinct and range from 1 to 3. These matrices, and the form of the wavefunction, have a deep mathematical significance. The algebraic structure represented by the Dirac matrices had been created some 50 years earlier by the English mathematician W. K. Clifford, which in turn had been based on the mid-19th century work of the German mathematician Hermann Grassmann in his "Lineale Ausdehnungslehre" (Theory of Linear Extensions). The latter had been regarded as well-nigh incomprehensible by most of his contemporaries. The appearance of something so seemingly abstract, at such a late date, in

Quantum physics

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

Quantum mechanics

Introduction to...
 Mathematical formulation of...
Fundamental concepts

Decoherence · Interference
 Uncertainty · Exclusion
 Transformation theory
 Ehrenfest theorem · Measurement
 Superposition · Entanglement

Experiments

Double-slit experiment
 Davisson-Germer experiment
 Stern–Gerlach experiment
 Bell's inequality experiment
 Popper's experiment
 Schrödinger's cat

Equations

Schrödinger equation
 Pauli equation
 Klein-Gordon equation

Dirac equation

Advanced theories

Quantum field theory
 Wightman axioms
 Quantum electrodynamics
 Quantum chromodynamics
 Quantum gravity
 Feynman diagram

Interpretations

Copenhagen · Ensemble
 Hidden variables · Transactional
 Many-worlds · Consistent histories
 Quantum logic
 Consciousness causes collapse

Scientists

Planck · Schrödinger
 Heisenberg · Bohr · Pauli
 Dirac · Bohm · Born
 de Broglie · von Neumann
 Einstein · Feynman
 Everett · Others

such a direct physical manner, amounts to one of the most remarkable chapters in the history of physics.

Comparison with the Schrodinger Equation

The Dirac equation is superficially similar to the Schrodinger equation for a free mass:

$$-\frac{\hbar^2}{2m}\nabla^2\phi = i\hbar\frac{\partial}{\partial t}\phi$$

The left side represents the square of the momentum operator divided by twice the mass, which, classically speaking, is the kinetic energy. If one wants to get a relativistic generalization of this equation, then the space and time derivatives must enter symmetrically, as they do in the Maxwell theory of the electromagnetic field, which is known to be relativistically invariant - that is, the derivatives must be of the *same order* in space and time. Now, in relativity, the momentum and the energy are each part of an invariant object, the 4-momentum, and they are connected by the relativistically invariant relation

$$\frac{E^2}{c^2} - p^2 = m^2c^2$$

with m now representing the rest mass. If we replace E and p by their operator equivalents in the Schrodinger theory, we get a differential equation that is a valid relativistic generalization of the Schrodinger equation:

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\phi = \frac{m^2c^2}{\hbar^2}\phi$$

where it is assumed that the wave function is now a relativistic scalar. In fact Schrodinger, who was well acquainted with relativity, tried this equation *before* the one that bears his name, but found it unsuitable. Because the time derivative is second order, one must specify both the initial value of $\partial_t\phi$ as well as ϕ itself when solving the equation. This is typical in the solution of problems of wave propagation, as in electrodynamics. However, in quantum theory, one is interested not in the actual motion as such, rather, the *energy spectrum* - mathematically, what is needed is a well-defined *eigenvalue problem*. As in electrodynamics, there will be advanced waves that appear to be propagating backward in time toward the source - these can be safely discarded as unphysical in electrodynamics, but *not here*, because one needs *all* the solutions in order to be able to express any solution as an expansion in terms of energy eigenfunctions and the corresponding eigenvalues.

There was an even more serious objection to be raised - in the Schrödinger theory, the probability density is given by the positive definite expression

$$\rho = \phi^* \phi$$

and its current by

$$J = -\frac{i\hbar}{2m}(\phi^*\nabla\phi - \phi\nabla\phi^*)$$

with the conservation of probability density expressed as

$$\nabla \cdot J + \frac{\partial\rho}{\partial t} = 0$$

In a relativistic theory, the form of the probability density must match that of the current when we replace ∇ by ∂_t , and in order that the conservation of probability current be a relativistically invariant expression, must form the 0-component of a 4-vector - thus we must have

$$\rho = \frac{i\hbar}{2m}(\phi^*\partial_t\phi - \phi\partial_t\phi^*)$$

Everything is now perfectly relativistic, but *the probability density is not positive definite*, because one may freely choose the initial values of both ϕ and $\partial_t\phi$. Such a theory would not have a simple, immediate physical interpretation, and so Schrodinger

abandoned it. (Though it was short-lived as a single-particle equation, it is resurrected in quantum field theory, where it is known as the Klein-Gordon equation, and describes particles of spin-0.)

Dirac's Coup

What is needed, then, is an equation that is first-order in both space and time. One could formally take the relativistic expression for the energy $E = c\sqrt{p^2 + m^2c^2}$, replace p by its operator equivalent, expand the square root in an infinite series of derivative operators, set up an eigenvalue problem, then solve the equation formally by iterations. Most physicists had little faith in such a process, even if it were technically possible.

As the story goes, Dirac was staring into the fireplace at Cambridge, pondering this problem, when he hit upon the idea of taking the square root of the wave operator thus:

$$\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} = (A\partial_x + B\partial_y + C\partial_z + \frac{i}{c}D\partial_t)(A\partial_x + B\partial_y + C\partial_z + \frac{i}{c}D\partial_t)$$

On multiplying out the right side, we see that in order to get all the cross-terms such as $\partial_x\partial_y$ to vanish, we must assume

$$AB + BA = 0, \dots$$

with

$$A^2 = B^2 = \dots = 1$$

Dirac, who had just then been intensely involved with working out the foundations of Heisenberg's matrix mechanics, immediately understood that these conditions could be met if A, B, \dots are *matrices*, with the implication that the wave function has *multiple components*. This immediately explained the appearance of two-component wave functions in Pauli's phenomenological theory of spin, something that up until then had been regarded as mysterious, even to Pauli himself. However, one needs at least 4x4 matrices to set up a system with the properties desired - so the wave function had *four* components, not two, as in the Pauli theory.

Given the factorization in terms of these matrices, one can now write down immediately an equation

$$(A\partial_x + B\partial_y + C\partial_z + \frac{i}{c}D\partial_t)\psi = \kappa\psi$$

with κ to be determined. Applying again the matrix operator on either side yields

$$(\nabla^2 - \frac{1}{c^2}\partial_t^2)\psi = \kappa^2\psi$$

On taking $\kappa = \frac{mc}{\hbar}$ we find that all the components of the wave function *individually* satisfy the relativistic energy-momentum relation. Thus the sought-for equation that is first-order in both space and time is

$$(A\partial_x + B\partial_y + C\partial_z + \frac{i}{c}D\partial_t - \frac{mc}{\hbar})\psi = 0$$

With $(A, B, C) = i\beta\alpha_k$ and $D = \beta$, we get the Dirac equation.

Comparison with the Pauli Theory

The necessity of introducing half-integral spin goes back experimentally to the results of the Stern-Gerlach experiment. A beam of atoms is run through a strong inhomogeneous magnetic field, which then splits into N parts depending on the intrinsic angular momentum of the atoms. It was found that for silver atoms, the beam was split in two - the ground state therefore could not be integral, because even if the intrinsic angular momentum of the atoms were as small as possible, 1, the beam would be split into 3 parts, corresponding to atoms with $L_z = -1, 0, \text{ and } +1$. The conclusion is that silver atoms have net intrinsic angular momentum

of 1/2. Pauli set up a theory which explained this splitting by introducing a two-component wave function and a corresponding correction term in the Hamiltonian, representing a semi-classical coupling of this wave function to an applied magnetic field, as so:

$$H = \frac{1}{2m}(\boldsymbol{\sigma} \cdot (\mathbf{p} - \frac{e}{c}\mathbf{A}))^2 + eA^0$$

Here A^μ is the applied electromagnetic field, and the three sigmas are Pauli matrices. e is the charge of the particle, e.g. $e = -e_0$ for the electron. On squaring out the first term, a residual interaction with the magnetic field is found, along with the usual Hamiltonian of a charged particle interacting with an applied field:

$$H = \frac{1}{2m}(\mathbf{p} - \frac{e}{c}\mathbf{A})^2 + eA^0 - \frac{e\hbar}{2mc}\boldsymbol{\sigma} \cdot \mathbf{B}$$

This Hamiltonian is now a 2x2 matrix, so the Schrödinger equation based on it,

$$H\phi = i\hbar\frac{\partial\phi}{\partial t}$$

must use a two-component wave function. Pauli had introduced the sigma matrices

$$\sigma_k = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as pure *phenomenology* - Dirac now had a *theoretical argument* that implied that spin was somehow the consequence of the marriage of quantum theory to relativity.

The Pauli matrices share the same properties as the Dirac matrices - they are all Hermitian, square to 1, and anticommute. This allows one to immediately find a representation of the Dirac matrices in terms of the Pauli matrices:

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}$$

The Dirac equation now may be written as an equation coupling two-component spinors:

$$\begin{pmatrix} mc^2 & c\boldsymbol{\sigma} \cdot \mathbf{p} \\ c\boldsymbol{\sigma} \cdot \mathbf{p} & -mc^2 \end{pmatrix} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} = i\hbar\frac{\partial}{\partial t} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}$$

Notice that on the diagonal we find the rest energy of the particle. If we set the momentum to zero - that is, bring the particle to rest - then we have

$$i\hbar\frac{\partial}{\partial t} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} = \begin{pmatrix} mc^2 & 0 \\ 0 & -mc^2 \end{pmatrix} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}$$

The equations for the individual two-spinors are now decoupled, and we see that the "top" and "bottom" two-spinors are individually eigenfunctions of the energy with eigenvalues equal to plus and minus the rest energy, respectively. The appearance of this negative eigenvalue begins to reveal the new physical content in the Dirac equation - states of *negative energy*. At first, this seems like a disaster similar to the one that plagued the second-order equation. It is not, and the physical interpretation of these states leads to the idea of *antimatter*. The Dirac equation predicts a *new phase of matter*. Its existence was subsequently confirmed in the laboratory. Like Newton's calculation of the Moon's orbital period and Einstein's prediction of the deflection of starlight, this is one of the great triumphs in the history of science.

It should be strongly emphasized that this separation in the rest frame is *not* an invariant statement - the "bottom" two-spinor

does not represent antimatter as such in general. The entire four-component spinor represents an *irreducible* whole - in general, states will have an admixture of positive *and* negative energy components. If we couple the Dirac equation to an electromagnetic field, as in the Pauli theory, then the positive and negative energy parts will be mixed together, even if they are originally decoupled. Dirac's main problem was to find a consistent interpretation of this mixing. As we shall see below, it brings a new phenomenon into physics - matter/antimatter creation and annihilation.

Covariant form and Relativistic Invariance

The covariant form of the Dirac equation is (employing the Einstein summation convention)

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\psi = 0$$

In the above, γ^0 is Hermitian, and the γ^k are anti-Hermitian, with the definition

$$\begin{aligned}\gamma^0 &= \beta \\ \gamma^k &= \gamma^0\alpha_k\end{aligned}$$

This may be summarized using the Minkowski metric on spacetime in the form

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

where the bracket expression $\{a, b\}$ means $ab + ba$, the anticommutator. These are the defining relations of a Clifford algebra over a pseudo-orthogonal 4-d space with metric signature $(+---)$. Note that one may also employ the metric form $(-+++)$ by multiplying all the gammas by a factor of i . At an elementary level, the choice may be regarded as conventional, but there are specific reasons for preferring the former, both mathematically and for convenience in calculation and physical interpretation. In the literature, one almost always finds the convention $(+---)$ in use. The specific Clifford algebra employed in the Dirac equation is known as the Dirac algebra.

The Dirac equation may be interpreted as an eigenvalue expression, where the rest mass is proportional to an eigenvalue of the 4-momentum operator, the proportion being the speed of light in vacuo:

$$P_{op}\psi = mc\psi$$

In practice, physicists often use units of measure such that \hbar and c are equal to 1, known as "natural" units. The equation is then multiplied through by $-i$ and takes the simple form

$$(\gamma^\mu\partial_\mu + im)\psi = 0$$

A fundamental theorem states that if two distinct sets of matrices are given that both satisfy the Clifford relations, then they are connected to each other by a similarity transformation:

$$\gamma'^\mu = S^{-1}\gamma^\mu S$$

If in addition the matrices are all unitary, as are the Dirac set, then S itself is unitary;

$$\gamma'^\mu = U^\dagger\gamma^\mu U$$

The transformation U is unique up to a multiplicative factor of absolute value 1. Let us now imagine a Lorentz transformation to have been performed on the derivative operators, which form a covariant vector. In order that the operator $\gamma^\mu\partial_\mu$ remain invariant, the gammas must transform among themselves as a contravariant vector with respect to their spacetime index. These new gammas will themselves satisfy the Clifford relations, because of the orthogonality of the Lorentz transformation. By the fundamental theorem, we may replace the new set by the old set subject to a unitary transformation. In the new frame, remembering that the rest mass is a relativistic scalar, the Dirac equation will then take the form

$$\begin{aligned}(U^\dagger\gamma^\mu U\partial'_\mu + im)\psi(x', t') &= 0 \\ U^\dagger(\gamma^\mu\partial'_\mu + im)U\psi(x', t') &= 0\end{aligned}$$

If we now define the transformed spinor

$$\psi' = U\psi$$

then we have the transformed Dirac equation

$$(\gamma^\mu \partial'_\mu + im)\psi'(x', t') = 0$$

Thus, once we settle on a unitary representation of the gammas, it is final providing we transform the spinor according the unitary transformation that corresponds to the given Lorentz transformation.

These considerations reveal the origin of the gammas in *geometry*, hearkening back to Grassmann's original motivation - they represent a fixed basis of unit vectors in spacetime. Similarly, products of the gammas such as $\gamma_\mu \gamma_\nu$ represent *oriented surface elements*, and so on. With this in mind, we can find the form the unit volume element on spacetime in terms of the gammas as follows. By definition, it is

$$V = \frac{1}{4!} \epsilon_{\mu\nu\alpha\beta} \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta$$

In order that this be an invariant, the epsilon symbol must be a tensor, and so must contain a factor of \sqrt{g} , where g is the determinant of the metric tensor. Since this is negative, that factor is *imaginary*. Thus

$$V = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$$

This matrix is given the special symbol γ_5 , owing to its importance when one is considering improper transformations of spacetime, that is, those that change the orientation of the basis vectors. In the representation we are using for the gammas, it is

$$\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Also note that could as easily have taken the negative square root of the determinant of g - the choice amounts to an initial handedness convention.

Adjoint Equation and Dirac Current

By defining the adjoint spinor

$$\bar{\psi} = \psi^\dagger \gamma^0$$

and noticing that

$$(\gamma^\mu)^\dagger \gamma^0 = \gamma^0 \gamma^\mu,$$

we obtain, by taking the Hermitian conjugate of the Dirac equation and multiplying from the right by γ^0 , the adjoint equation:

$$\bar{\psi}(\gamma^\mu \partial_\mu - im) = 0$$

where ∂_μ is understood to act to the left. Multiplying the Dirac equation by $\bar{\psi}$ from the left, and the adjoint equation by ψ from the right, and adding, produces the law of conservation of the Dirac current in covariant form:

$$\partial_\mu (\bar{\psi} \gamma^\mu \psi) = 0$$

Now we see the great advantage of the first-order equation over the one Schrödinger had tried - this is the conserved probability current density required by relativistic invariance, only now its 0-component is *positive definite*:

$$J^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi$$

The Dirac equation and its adjoint are the Euler-Lagrange equations of the 4-d invariant action integral

$$S = \int L d^4\omega$$

where the scalar L is the Dirac Lagrangian

$$L = mc\bar{\psi}\psi - \frac{i\hbar}{2}(\bar{\psi}\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma^\mu\psi)$$

and for the purposes of variation, ψ and $\bar{\psi}$ are regarded as independent fields. The relativistic invariance also follows immediately from the variational principle.

Coupling to an Electromagnetic Field

To consider problems in which an applied electromagnetic field interacts with the particles described by the Dirac equation, one uses the correspondence principle, and takes over into the theory the corresponding expression from classical mechanics, whereby the total momentum of a charged particle in an external field is modified as so:

$$p \rightarrow p - \frac{e}{c}A$$

In natural units, the Dirac equation then takes the form

$$(\gamma^\mu(\partial_\mu + ieA_\mu) + im)\psi = 0$$

This validity of this prescription is confirmed experimentally with great precision. It is known as *minimal coupling*, and is found throughout particle physics. Indeed, while the introduction of the electromagnetic field in this way is essentially phenomenological in this context, it rises to a fundamental principle in quantum field theory.

Now as stated above, the transformation U is defined only up to a phase factor $e^{i\theta}$. Also, the fundamental observable of the Dirac theory, the current, is unchanged if we multiply the wave function by an arbitrary phase. We may exploit this to get the form of the mutual interaction of a Dirac particle and the electromagnetic field, as opposed to simply considering a Dirac particle in an applied field, by assuming this arbitrary phase factor to depend continuously on position:

$$\psi \rightarrow \psi' = e^{i\theta(x,t)}\psi$$

Notice now that

$$\gamma^\mu(\partial_\mu + ieA_\mu)\psi' = e^{i\theta}\gamma^\mu(\partial_\mu + ie(A_\mu + \frac{1}{e}\partial_\mu\theta))\psi = e^{i\theta}\gamma^\mu(\partial_\mu + ieA'_\mu)\psi$$

In order to preserve minimal coupling, we must add to the potential a term proportional to the gradient of the phase. But we know from electrodynamics that this leaves the electromagnetic field itself invariant. The value of the phase is arbitrary, but *not* how it changes from place to place. This is the starting point of gauge theory, which is the main principle on which quantum field theory is based. The simplest such theory, and the one most thoroughly understood, is known as quantum electrodynamics. The equations of field theory thus have invariance under both Lorentz transformations and gauge transformations.

Physical Interpretation

The Dirac theory, while providing a wealth of information that is accurately confirmed by experiments, nevertheless introduces a new physical paradigm that appears at first difficult to interpret and even paradoxical. Some of these issues of interpretation must be regarded as open questions. Here we will see how the Dirac theory brilliantly answered some of the outstanding issues in physics at the time it was put forward, while posing others that are still the subject of debate.

Identification of Observables

The critical physical question in a quantum theory is - what are the physically observable quantities defined by the theory? According to general principles, such quantities are defined by Hermitian operators that act on the Hilbert space of possible states of a system. The eigenvalues of these operators are then the possible results of measuring the corresponding physical quantity. In the Schrödinger theory, the simplest such object is the overall Hamiltonian, which represents the total energy of the system. If we wish to maintain this interpretation on passing to the Dirac theory, we must take the Hamiltonian to be

$$H = \gamma^0 \left(mc^2 + c \sum_{k=1}^3 \gamma^k \left(p_k - \frac{e}{c} A_k \right) c \right) + eA^0$$

This looks promising, because we see by inspection the rest energy of the particle and, in case $A = 0$, the energy of a charge placed in an electric potential eA^0 . What about the term involving the vector potential? In classical electrodynamics, the energy of a charge moving in an applied potential is

$$H = c \sqrt{\left(p - \frac{e}{c} A \right)^2 + m^2 c^2} + eA^0$$

Thus the Dirac Hamiltonian is *fundamentally distinguished* from its classical counterpart, and we must take great care to correctly identify what is an observable in this theory. Much of the apparent paradoxical behavior implied by the Dirac equation amounts to a misidentification of these observables. Let us now describe one such effect. (cont'd)

History

Since the Dirac equation was originally invented to describe the electron, we will generally speak of "electrons" in this article. The equation also applies to quarks, which are also elementary spin- $\frac{1}{2}$ particles. A modified Dirac equation can be used to approximately describe protons and neutrons, which are not elementary particles (they are made up of quarks), but have a net spin of $\frac{1}{2}$. Another modification of the Dirac equation, called the Majorana equation, is thought to describe neutrinos — also spin- $\frac{1}{2}$ particles.

The Dirac equation describes the probability amplitudes for a *single* electron. This is a single-particle theory; in other words, it does not account for the creation and destruction of the particles. It gives a good prediction of the magnetic moment of the electron and explains much of the fine structure observed in atomic spectral lines. It also explains the spin of the electron. Two of the four solutions of the equation correspond to the two spin states of the electron. The other two solutions make the peculiar prediction that there exist an infinite set of quantum states in which the electron possesses negative energy. This strange result led Dirac to predict, via a remarkable hypothesis known as "hole theory," the existence of particles behaving like positively-charged electrons. Dirac thought at first these particles might be protons. He was chagrined when the strict prediction of his equation (which actually specifies particles of the same mass as the electron) was verified by the discovery of the positron in 1932. When asked later why he hadn't actually boldly predicted the yet unbound positron with its correct mass, Dirac answered "Pure cowardice!" He shared the Nobel Prize anyway, in 1933.

Despite these successes, Dirac's theory is flawed by its neglect of the possibility of creating and destroying particles, one of the basic consequences of relativity. This difficulty is resolved by reformulating it as a quantum field theory. Adding a quantized electromagnetic field to this theory leads to the theory of quantum electrodynamics (QED). Moreover the equation cannot fully account for particles of negative energy but is restricted to positive energy particles.

A similar equation for spin $3/2$ particles is called the Rarita-Schwinger equation.

Hole theory

The negative E solutions found in the preceding section are problematic, for it was assumed that the particle has a positive energy. Mathematically speaking, however, there seems to be no reason for us to reject the negative-energy solutions. Since they exist, we cannot simply ignore them, for once we include the interaction between the electron and the electromagnetic field, any

electron placed in a positive-energy eigenstate would decay into negative-energy eigenstates of successively lower energy by emitting excess energy in the form of photons. Real electrons obviously do not behave in this way.

To cope with this problem, Dirac introduced the hypothesis, known as **hole theory**, that the vacuum is the many-body quantum state in which all the negative-energy electron eigenstates are occupied. This description of the vacuum as a "sea" of electrons is called the Dirac sea. Since the Pauli exclusion principle forbids electrons from occupying the same state, any additional electron would be forced to occupy a positive-energy eigenstate, and positive-energy electrons would be forbidden from decaying into negative-energy eigenstates.

Dirac further reasoned that if the negative-energy eigenstates are incompletely filled, each unoccupied eigenstate – called a **hole** – would behave like a positively charged particle. The hole possesses a *positive* energy, since energy is required to create a particle–hole pair from the vacuum. As noted above, Dirac initially thought that the hole might be the proton, but Hermann Weyl pointed out that the hole should behave as if it had the same mass as an electron, whereas the proton is over 1800 times heavier. The hole was eventually identified as the positron, experimentally discovered by Carl Anderson in 1932.

It is not entirely satisfactory to describe the "vacuum" using an infinite sea of negative-energy electrons. The infinitely negative contributions from the sea of negative-energy electrons has to be canceled by an infinite positive "bare" energy and the contribution to the charge density and current coming from the sea of negative-energy electrons is exactly canceled by an infinite positive "jellium" background so that the net electric charge density of the vacuum is zero. In quantum field theory, a Bogoliubov transformation on the creation and annihilation operators (turning an occupied negative-energy electron state into an unoccupied positive energy positron state and an unoccupied negative-energy electron state into an occupied positive energy positron state) allows us to bypass the Dirac sea formalism even though, formally, it is equivalent to it.

In certain applications of condensed matter physics, however, the underlying concepts of "hole theory" are valid. The sea of conduction electrons in an electrical conductor, called a Fermi sea, contains electrons with energies up to the chemical potential of the system. An unfilled state in the Fermi sea behaves like a positively-charged electron, though it is referred to as a "hole" rather than a "positron". The negative charge of the Fermi sea is balanced by the positively-charged ionic lattice of the material.

Dirac bilinears

There are five different (neutral) Dirac bilinear terms not involving any derivatives:

- (S)calar: $\bar{\psi}\psi$ (scalar, P-even)
- (P)seudoscalar: $\bar{\psi}\gamma^5\psi$ (scalar, P-odd)
- (V)ector: $\bar{\psi}\gamma^\mu\psi$ (vector, P-even)
- (A)xial: $\bar{\psi}\gamma^\mu\gamma^5\psi$ (vector, P-odd)
- (T)ensor: $\bar{\psi}\sigma^{\mu\nu}\psi$ (antisymmetric tensor, P-even),

where $\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$ and $\gamma^5 = \gamma_5 = \frac{i}{4!} \epsilon_{\mu\nu\rho\lambda} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\lambda = i\gamma^0\gamma^1\gamma^2\gamma^3$.

A Dirac mass term is an S coupling. A Yukawa coupling may be S or P. The electromagnetic coupling is V. The weak interactions are V-A.

Dirac Equation in Curved Spacetime

The Dirac equation as stated above is a relativistic quantum mechanical wave equation invented by Paul Dirac in 1928 (Dirac 1928a, 1928b) originally designed to overcome the criticism of the Klein-Gordon Equation. The Klein-Gordon equation gave negative probabilities and this is considered to be physically meaningless. Despite this fact, this equation accounts well for Bosons, that is spin zero particles. This criticism leveled against the Klein-Gordon equation, motivated Dirac to seek an equation devoid of negative probabilities.

The Dirac equation, as demonstrated above, is consistent with Quantum Mechanics (QM) and also fully consistent with the Special Theory of Relativity. The equation accounts in a natural way for the nature of particle spin as a relativistic phenomenon. The equation successfully predicted the existence of anti-particles. In its bare form, the equation provides an impressive and accurate description of the Electron hence it being referred in most of the literature as the "Dirac Equation for the Electron". It also accounts well for quarks and other spin half particles although in some of the cases, there is need for slight modifications -

for example, you need the Proca equation to describe the neutron.

The first taste of glory of the Dirac equation was it being able to account in a natural way for the gyromagnetic ratio of the electron, that is $g = 2$, which can not be accounted for using non-relativistic QM. For several years after its discovery, most physicists believed that it also described the Proton and the Neutron, which are both spin $1/2$ particles. In simple terms, it was thought or presumed that the Dirac equation was a universal equation for spin $1/2$ particles.

However, beginning with the experiments of Otto Stern and Otto Frisch in 1933, the magnetic moments of these particles were found to disagree significantly with the predictions of the Dirac equation. The Proton was found to have a gyromagnetic ratio $g_p = 5.58$ which is 2.79 times larger than that predicted by the Dirac equation. The Neutron, which is electrically neutral spin- $1/2$ particle was found to have a gyromagnetic ratio $g_n = -3.83$.

These anomalous magnetic moments of the Neutron and Proton which are clearly not conformatory to the Dirac Equation have been taken to be experimental indication that these particles are not fundamental particles. In the case of the Neutron, yes it is clearly not a fundamental particle since it does decay into a Proton, Electron and Neutrino, that is, $n \rightarrow p + e^- + \bar{\nu}_e$. If the Dirac equation is a universal equation for fundamental fermion particles, then any fundamental fermion particle must conform to this equation. Equivalently, any spin $1/2$ particle that can not be described by it, must therefore not be a fundamental particle of nature. By definition a fundamental particle is a particle known to have no sub-structure, that is, it can not be broken down into smaller particles that will not decay into anything else.

From the Standard Model, we know that the Proton and Neutron are composed of quarks thus are not fundamental particles. The question is; is this the reason why these particle's gyromagnetic ratio is different from that predicted by the bare Dirac equation? Prevailing wisdom suggests that the anomalous gyromagnetic ratio arises because the particles under question are not fundamental particles. From the theory of the Dirac Equation in Curved Spacetime, the answer to this is a clear no. The reason for the deviation of the gyromagnetic ratio from that expected from the Dirac theory is (according to this theory) because particles do have a finite size and that spacetime is curved. In this theory, the anomalous gyromagnetic ratio arises from the interaction of spin with the Lorentz force in a curved spacetime for a particle of finite spatial size and mass. The derived relation for the anomalous magnetic moment and the particle size is similar to that deduced by Brodsky & Drell (1980) and experimentally verified by Dehmelt (1989). Brodsky & Drell (1980) proposed that fermions do have a sub-structure and this gives rise to the anomalous gyromagnetic ratio which varies as the spatial size and inverse to the mass.

The question has always been there, what is the curved spacetime version of the Dirac Equation? Many proposals have been put forward. A new equation has come forth that tries to describe the evolution of the Dirac spinor in curved spacetime. This equation is

$$i\hbar\bar{\gamma}^\mu\partial_\mu\psi - m_0c\psi = 0$$

and its symbolic form makes it look more like the normal Dirac Equation and the subtleness is found in that the $\bar{\gamma}^\mu$ are not the same as the 4×4 Dirac gamma matrices. Call the matrices $\bar{\gamma}^\mu$ the Dirac bar-gamma matrices -- these matrices are given

$$\bar{\gamma}^\mu = \begin{bmatrix} I & i\sqrt{2}\sigma^\mu \\ i\sqrt{2}\sigma^\mu & -I \end{bmatrix},$$

where the σ^μ for $\mu = 1,2,3$ are the usual Pauli matrices and for the case $\mu = 0$ this is the 2×2 null matrix and I is the 2×2 identity matrix. As already pointed out, notice that the Dirac bar-gamma matrices are different from the normal Dirac 4×4 matrices. This equation predicts that the gyromagnetic ratio of fermions will differ from $g = 2$ which is predicted by the bare Dirac equation which is applicable to curved spacetime. The predicted anomalous gyromagnetic ratio by this equation is

$$\Delta g = \left(\frac{g-2}{2}\right) = \sqrt{2} \left(1 \pm \frac{\lambda_c}{R_p}\right) - 1,$$

where R_p is the radius of the particle, λ_c is the compton wavelength of the particle and \pm depends on the particle's spin relative to the magnetic field. For more details on this equation see [1].

See also

- Breit equation

- Klein-Gordon equation
- Quantum electrodynamics
- Rarita-Schwinger equation
- Feynman checkerboard
- Theoretical and experimental justification for the Schrödinger equation

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