Kronecker delta

From Wikipedia, the free encyclopedia

In mathematics, the **Kronecker delta** or **Kronecker's delta**, named after Leopold Kronecker (1823-1891), is a function of two variables, usually integers, which is 1 if they are equal, and 0 otherwise. So, for example, $\delta_{12} = 0$, but $\delta_{33} = 1$. It is written as the symbol δ_{ij}, and treated as a notational shorthand rather than as a function.

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Contents

Alternate notation

Using the Iverson bracket:

$$\delta_{ij} = [i = j]$$

Often, the notation δ_i is used.

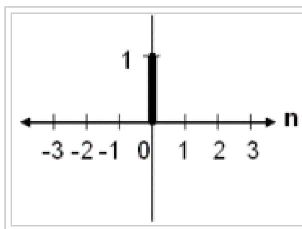
$$\delta_i = \begin{cases} 1, & \text{if } i = 0 \\ 0, & \text{if } i \neq 0 \end{cases}$$

In linear algebra, it can be thought of as a tensor, and is written δ^i_j.

Digital signal processing

Similarly, in digital signal processing, the same concept is represented as a function on $\mathbb{Z}$ (the integers)**:**

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



An impulse function

The function is referred to as an *impulse*, or *unit impulse*. And when it stimulates a signal processing element, the output is called the impulse response of the element.

Properties of the delta function

The Kronecker delta has the so-called *sifting* property that for $j \in \mathbb{Z}$:

$$\sum_{i=-\infty}^{\infty} \delta_{ij} a_i = a_j.$$

and if the integers are viewed as a measure space, endowed with the counting measure, then this property coincides with the defining property of the Dirac delta function

$$\int_{-\infty}^{\infty} \delta(x-y)f(x)dx = f(y),$$

and in fact Dirac's delta was named after the Kronecker delta because of this analogous property. In signal processing it is usually the context (discrete or continuous time) that distinguishes the Kronecker and Dirac "functions". And by convention, $\delta(t)$ generally indicates continuous time (Dirac), whereas arguments like *i*, *j*, *k*, *l*, *m*, and *n* are usually reserved for discrete time (Kronecker). Another common practice is to represent discrete sequences with square brackets; thus: $\delta[n]$. It is important to note that the Kronecker delta is not the result of sampling the Dirac delta function.

The Kronecker delta is used in many areas of mathematics.

Linear algebra

In linear algebra, the identity matrix can be written as δ_{ij}.

If it is considered as a tensor, the **Kronecker tensor**, it can be written δ^i_j with a covariant index *i* and contravariant index *j*.

This (1,1) tensor represents:

- the identity matrix, considered as a linear mapping
- the trace
- the inner product $V^* \otimes V \rightarrow K$
- the map $K \rightarrow V^* \otimes V$, representing scalar multiplication as a sum of outer products

Extensions of the delta function

In the same fashion, we may define an analogous, multi-dimensional function of many variables

$$\delta^{j_1 j_2 \dots j_n}_{i_1 i_2 \dots i_n} = \prod_{k=1}^{n} \delta_{i_k j_k}.$$

This function takes the value 1 if and only if all the upper indices match the corresponding lower ones, and the value zero otherwise.

Integral representations

For any integer *n*, using a standard residue calculation we can write an integral representation for the Kronecker delta as

$$\delta_{x,n} = \frac{1}{2\pi i} \oint z^{x-n-1} dz,$$

where the contour of the integral goes counterclockwise around zero. This represention is also equivalent to

$$\delta_{x,n} = \frac{1}{2\pi} \int_0^{2\pi} e^{i(x-n)\varphi} d\varphi,$$

by a rotation in the complex plane.

See also

- Levi-Civita symbol
- Dirac delta function

- Dirac measure

Categories: Mathematical notation | Tensors