Equality (mathematics)

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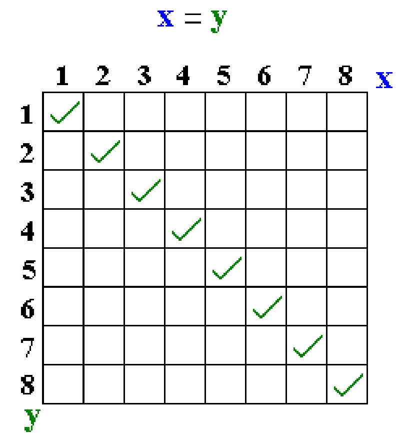

Two mathematical objects are **equal** if and only if they are precisely the same in every way. The complementary notion is distinctness. This defines a binary relation, **equality**, denoted by the **sign of equality** "=" in such a way that the statement "$x = y$" means that x and y are equal.

Equality is the paradigmatic example of the more general concept of equivalence relations on a set: those binary relations that are reflexive, symmetric, and transitive. It goes beyond the other equivalence relations by also being antisymmetric. In fact, these four properties uniquely determine the equality relation on any set S and render equality the only relation on S that is both an equivalence relation and a partial order. It follows from this that equality is the smallest equivalence relation on any set S, in the sense that it is a subset of any other equivalence relation on S.

An equation is simply an assertion that two expressions are related by equality.

Beware that the symbol "=" is sometimes used for relations other than equality. For example, the statement $T(n) = O(n^2)$ means that $T(n)$ grows at the *order* of n^2. Despite the notation, the statement is actually better understood as asserting a set membership: $O(f(n))$ is formally the set of all functions on the positive integers that, for large n, grow no faster than $f(n)$. In particular, since membership, unlike equality, is not symmetric, it is meaningless to write $O(n^2) = T(n)$. See Big O notation for more on this.

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Logical formulations

The equality relation is always defined such that things that are equal have all and only the same properties. Often equality is just defined as identity.

A stronger sense of equality is obtained if some form of Leibniz's law is added as an axiom; the assertion of this axiom rules out "bare particulars"—things that have all and only the same properties but are not equal to each other—which are possible in some logical formalisms. The axiom states that two things are equal if they have all and only the same properties. Formally:

> Given any x and y, $x = y$ if, given any predicate P, $P(x)$ if and only if $P(y)$.

In this law, the connective "if and only if" can be weakened to "if"; the modified law is equivalent to the original.

Instead of considering Leibniz's law as an axiom, it can also be taken as the *definition* of equality. The property of being an equivalence relation, as well as the properties given below, can then be proved: they become theorems.

Some basic logical properties of equality

The substitution property states:

- For any quantities a and b and any expression $F(x)$, if $a = b$, then $F(a) = F(b)$ (if either side makes sense).

In first-order logic, this is a schema, since we can't quantify over expressions like F (which would be a functional predicate).

Some specific examples of this are:

- For any real numbers a, b, and c, if $a = b$, then $a + c = b + c$ (here $F(x)$ is $x + c$);
- For any real numbers a, b, and c, if $a = b$, then $a - c = b - c$ (here $F(x)$ is $x - c$);
- For any real numbers a, b, and c, if $a = b$, then $ac = bc$ (here $F(x)$ is xc);
- For any real numbers a, b, and c, if $a = b$ and c is not zero, then $a/c = b/c$ (here $F(x)$ is x/c).

The reflexive property states:

For any quantity a, $a = a$.

This property is generally used in mathematical proofs as an intermediate step.

The symmetric property states:

- For any quantities a and b, if $a = b$, then $b = a$.

The transitive property states:

- For any quantities a, b, and c, if $a = b$ and $b = c$, then $a = c$.

The binary relation "is approximately equal" between real numbers or other things, even if more precisely defined, is not transitive (it may seem so at first sight, but many small differences can add up to something big). However, equality almost everywhere *is* transitive.

Although the symmetric and transitive properties are often seen as fundamental, they can be proved, if the substitution and reflexive properties are assumed instead.

References

- Cormen, Thomas H.; Leiserson, Charles E.; Rivest, Ronald L. (1990). *Introduction to Algorithms*, first edition, MIT Press and McGraw-Hill. ISBN 978-0-262-03141-7.

- Mac Lane, Saunders; Garrett Birkhoff (1967). *Algebra*. American Mathematical Society.

See also

- Equals sign
- Inequality
- Logical equality